MECHANICS.

205. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Given two points A and B not in the same horizontal nor in the same vertical line; to find the path from A to B along which a particle will slide from rest under the force of gravity alone so that the average velocity along the curve shall be a maximum.

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Let the particle start from rest at the point (x_1, y_1), where the axis of x is vertical and the axis of y horizontal. Then

$$v=\frac{ds}{dt}=\sqrt{[2g(x-x_1)]}=\frac{dx}{dt}\sqrt{1+\left(\frac{dy}{dx}\right)^2}.$$

Since v=a maximum, t=a minimum.

$$\therefore t=U=\int_{x_1}^{x_2}\frac{\sqrt{[1+(dy/dx)^2]}\,dx}{\sqrt{[2g(x-x_1)]}}=\text{minimum}.$$

By Calculus of Variations, since the expression in the right hand member does not contain y explicitly, the differential of this expression is equal to a constant.

$$\therefore \frac{dy/dx}{\sqrt{[2g(x-x_1)]}\sqrt{[1+(dy/dx)^2]}}=C.$$

$$\therefore \left(\frac{dy}{dx}\right)^2=C^2\left[1+\left(\frac{dy}{dx}\right)^2\right][2g(x-x_1)].$$

$$\therefore \frac{dy}{dx}=\pm\frac{C\sqrt{[2g(x-x_1)]}}{\sqrt{[1-2gC^2(x-x_1)]}},$$

the differential equation of a cycloid. This is Bernoulli's famous problem and is solved in almost every work on the Calculus of Variations as well as Dynamics.

206. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

A rigid square $ABDC$ made by smooth wires is fixed with A vertically above D. Two small equal spherical elastic beads slide down BD, CD, starting simultaneously from B and C. Find the ratio of their velocities of approach and separation at D, and how far they will separate after impact.

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

The particles are sliding down inclined planes, inclined at an angle $\beta=\frac{1}{4}\pi$ to the horizon, starting from rest. Let a=side of square, x=distance each particle has moved after a time t. Then the velocity of each is

$$v=\sqrt{(2g\sin\tfrac{1}{4}\pi.x)}=\sqrt{(gx\sqrt{2})}=\sqrt{(ga\sqrt{2})} \text{ at } D.$$

Each particle is moving toward the diagonal at a velocity $=v\sin\frac{1}{4}\pi=\frac{1}{2}v\sqrt{2}=\frac{1}{2}\sqrt{[2\sqrt{(2)}gx]}=\frac{1}{2}\sqrt{[2\sqrt{(2)}ga]}$ at D. Hence, the particles at D approach each other with a velocity $=2\times\frac{1}{2}\sqrt{[2\sqrt{(2)}ga]}=\sqrt{[2\sqrt{(2)}ga]}$.

Let e=the coefficient of restitution. Then, since the particles impinge at right angles, they will separate with a velocity $v_1=e\sqrt{[2\sqrt{(2)}ga]}$, and will ascend DB, DA, respectively, with a velocity $v_2=e\sqrt{(ga\sqrt{2})}$. Each will ascend a distance $=v_2^2/2g\sin\frac{1}{4}\pi=v_2^2/g\sqrt{2}=e^2a$.

Therefore, they separate, after impact, a distance $=2e^2a\sin\frac{1}{4}\pi=e^2a\sqrt{2}$.

If $e=1$, they return to their starting points.

AVERAGE AND PROBABILITY.

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190. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

A line is drawn at random across a regular $2n$-gon; what is the chance that it crosses parallel sides?

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Let AB, CD be two parallel sides. Through the center of the polygon O draw PQ to represent the direction of the random line.

Let $AO=r$, $\angle AOB=\pi/n=\beta$, $\angle AOP=\theta$.

For all possible positions PQ may be regarded as lying in one quadrant. The actual line parallel to PQ may hold any position within breadth of plane HF (H being the most remote vertex from QP, and F foot of perpendicular from H on QP) for all positions, and AE (A being the nearest vertex to QP and E foot of perpendicular from A to QP) for all favorable positions.

$\therefore$ The chance is $p=AE/HF$; $AE=r\sin\theta$, $HF=r\cos\theta$ for n even; $HF=r\cos(\frac{1}{2}\beta-\theta)$ for n odd. For n even,

$$p=\frac{2n}{\pi}\int_0^{\frac{1}{2}\beta}\tan\theta\,d\theta=\frac{2n}{\pi}\log(\sec\tfrac{1}{2}\beta). \quad \therefore p=\frac{2n}{\pi}\log\left(\sec\frac{\pi}{2n}\right).$$

For n odd, $p=\dfrac{2n}{\pi}\displaystyle\int_0^{\frac{1}{2}\pi}\frac{\sin\theta}{\cos(\frac{1}{2}\beta-\theta)}d\theta=\dfrac{n}{\pi}(\beta\sin\tfrac{1}{2}\beta+2\cos\tfrac{1}{2}\beta\log\cos\tfrac{1}{2}\beta)$.

$$\therefore p=\frac{n}{\pi}\left[\frac{\pi}{n}\sin\frac{\pi}{2n}+2\cos\frac{\pi}{2n}\log\left(\cos\frac{\pi}{2n}\right)\right].$$